ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 10, pp. 1186–1195.




# Parametrization of Optimal Anisotropic Controllers

**A. Yu. Kustov**

*Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
*e-mail: arkadiykustov@yandex.ru*

Received June 19, 2023
Revised July 18, 2023
Accepted August 2, 2023

**Abstract**—This paper provides a parametrization of optimal anisotropic controllers for linear discrete time invariant systems. The controllers to be designed are limited by causal dynamic output-feedback control laws. The obtained solution depends on several adjustable parameters that determine the specific type of controller, and is of the form of a system of the Riccati equations relating to a $\mathcal{H}_2$-optimal controller for a system formed by a series connection of the original system and the worst-case generating filter corresponding to the maximum value of the mean anisotropy of the external disturbance.

*Keywords*: linear discrete systems, anisotropy-based theory, optimal control, parametrization

**DOI:** 10.25728/arcRAS.2023.11.68.001

## 1. INTRODUCTION

The anisotropy-based control and estimation theory has been developed in the mid-90s as a response to attempts to provide the generalization of the results of the well-known $\mathcal{H}_2$- and $\mathcal{H}_\infty$-controller design theories [4, 12, 13].

It clearly shows the features of the control problems, the information theory, and various classical methods for suppressing (or mitigating) the impact of external disturbances [5]. However, unlike some approaches where it was proposed to use artificially defined in a certain sense mixed-type functionals, the anisotropy-based theory was focused on the method of describing the external disturbance driven the system. It was shown that the use of theoretical functionals makes it possible not only to describe a wide class of statistically uncertain random noises, but also generalize in a natural way the concepts of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms making them the limiting cases of the anisotropic norm.

In this paper, the problem of parametrization of optimal anisotropic controllers for linear discrete time invariant systems is solved. The solution to the problem is based on the result associated with the parametrization of $\mathcal{H}_2$-optimal controllers as well as with the equations for the worst-case generating filter used in the anisotropy-based theory to form a signal with a given threshold level of mean anisotropy.

The paper is organized as follows. In Section 2, some preliminary mathematics from anisotropy-based theory are given. It also contains a parametrization of the $\mathcal{H}_2$-optimal controllers. In Section 3, the problem of parametrization of optimal anisotropic controllers is solved. The results are demonstrated with a numerical example. The last section contains the conclusions.

## 2. PRELIMINARIES

### 2.1. Notations

$\mathcal{H}_2^{m\times n}$ is the Hardy space of analytic rational transfer functions $P(z) = \sum_{k=0}^{+\infty} P_k z^k \in \mathbb{C}^{m\times n}$ in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ having the finite $\mathcal{H}_2$-norm

$$\|P\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\left(\widehat{P}(\omega)\widehat{P}^{\mathrm{T}}(-\omega)\right)d\omega\right)^{1/2},$$

where $\widehat{P}(\omega) = \lim_{r\to 1-0} P(re^{i\omega})$; $\mathcal{RH}_2^{m\times n}$ is set of strictly proper stable rational $m\times n$ transfer functions; $\|P\|_\infty = \sup_{\omega\in[-2\pi;\pi)} \sigma_{\max}(\widehat{P}(\omega))$ is $\mathcal{H}_\infty$-norm of transfer matrix function $P(z)$ where $\sigma_{\max}(X) = \max_k \sigma_k(X)$ denotes maximum singular value of a matrix $X$, and $\sigma_k(X) = \lambda_k(X^{\mathrm{T}}X)$.

### 2.2. Basic Concepts of Anisotropy-Based Theory

Usually, the object of study in anisotropy-based theory is a stable linear discrete time invariant system

$$P_{zw} \sim \begin{cases} x_{k+1} = Ax_k + Bw_k, \\ z_k \quad\; = Cx_k + Dw_k, \end{cases} \tag{1}$$

with known matrices $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_w}$, $C \in \mathbb{R}^{n_z\times n_x}$, $D \in \mathbb{R}^{n_z\times n_w}$, and, in general, zero initial conditions ($x_0 = 0$). This system describes the relation between the dynamical processes $\{x_k\}_{k\geqslant 0}$ and $\{z_k\}_{k\geqslant 0}$ driven by random input disturbance $\{w_k\}_{k\geqslant 0}$. The system (1) corresponds to its transfer function $P_{zw}(z) = D + C(zI_{n_x} - A)^{-1}B$ given by the quadruple

$$P_{zw} \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]: \quad w \xrightarrow{x} z. \tag{2}$$

If necessary, we will specify the spaces of the states, the inputs, and the outputs. Within the discussion of the controller design problem, the plant (1) should be considered as the closed-loop system.

The following definitions give a basic idea of the concepts of the anisotropy-based theory. See [4, 12, 13] for more details.

**Definition 1.** The anisotropy of the square-integrable random vector $w \in \mathbb{L}_2^{n_w}$ is a nonzero number defined by

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{D}(f\|p_{n_w,\lambda}), \tag{3}$$

where $\mathbf{D}(f\|g)$ is the Kullback–Leibler information divergence of $f$ with respect to $g$; $f(x)$ is the probability density function (p.d.f.) of vector $w$; $p_{n_w,\lambda}(x) = (2\pi\lambda)^{-n_w/2}\exp\left(-\frac{|x|^2}{2\lambda}\right)$ is the p.d.f. of zero-mean Gaussian vector having scalar covariance matrix $\lambda I_{n_w}$.

**Definition 2.** The mean anisotropy of stationary ergodic random sequence $W = \{w_k\}_{k\geqslant 0}$ is defined by the following formula:

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{4}$$

Here, $W_{s:t} = (w_s^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}$ denotes the fragment of the sequence $W = \{w_k\}_{k\geqslant 0}$ for $k = s, s+1, \ldots, t-1, t$.

It is assumed that the system (1) is driven by a disturbance with mean anisotropy constrained by nonnegative number $a \geqslant 0$, i.e. $\overline{\mathbf{A}}(W) \leqslant a$. This limitation determines the ability of the nature to generate the worst-case (in the sense of the value of the root-mean-square (RMS) gain) external disturbance, which the $\mathcal{H}_\infty$-theory works with, but at the same time allows it to have both spatial and temporal correlations, which is not covered by the classic $\mathcal{H}_2$-theory.

**Definition 3.** Anisotropic norm of the system (1) driven by the input disturbance whose mean anisotropy satisfy $\overline{\mathbf{A}}(W) \leqslant a$ is defined as

$$|\!|\!| P_{zw} |\!|\!|_a = \sup \left\{ \frac{\|P_{zw}G\|_2}{\|G\|_2} : \ G \in \mathcal{H}_2^{n_w \times n_w} \wedge W = GV \wedge \overline{\mathbf{A}}(W) \leqslant a \right\} \tag{5}$$

where $V = \{v_k\}_{k \geqslant 0}$ denotes the standard Gaussian white noise passed through the linear system with $(n_w \times n_w)$-dimensional transfer function $G(z)$ having bounded $\mathcal{H}_2$-norm.

The anisotropic norm quantitatively reflects the ability of the system to amplify in the RMS sense the input signal with the information-theoretic constraint $\overline{\mathbf{A}}(W) \leqslant a$ imposed on it. In the case $\overline{\mathbf{A}}(W) = 0$, we have that $W = V$, and $|\!|\!| P_{zw} |\!|\!|_0 = \|P_{zw}\|_2/\sqrt{n_w}$. In the case when the restriction on mean anisotropy is removed, i.e. $\overline{\mathbf{A}}(W) < +\infty$, it can be shown that $\lim\limits_{a \to +\infty} |\!|\!| P_{zw} |\!|\!|_a = \|P_{zw}\|_\infty$. Thus, the anisotropy-based theory not only describes a wide class of external disturbances in information-theoretic terms, but also generalizes the approaches to controller design developed within the framework of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-theories.

### *2.3. Parametrization of $\mathcal{H}_2$-Optimal Controllers*

A lot of works are devoted to the study of all possible aspects of the behavior of linear systems with $\mathcal{H}_2$-optimal estimating controllers. In particular, a number of them describe methods for parameterizing the entire set of $\mathcal{H}_2$-optimal controllers. The procedure for solving this problem, as well as the accompanying difficulties, are described in detail in [2, 6, 9, 11] and many others. The main idea on which the solution is based is that $\mathcal{H}_2$-optimal controllers are directly related to the controllers that ensure the invariance of the output of some auxiliary system with respect to disturbances. Hence, by parameterizing the set of these controllers, parametrization of the $\mathcal{H}_2$-controllers can be obtained. The formulation of the problem of parametrization of $\mathcal{H}_2$-controllers, and the solution of this problem can be given in the following form.

Consider the system

$$F \sim \left[\begin{array}{c|cc} A & B_u & B_w \\ \hline C_y & 0 & D_{yw} \\ C_z & D_{zu} & 0 \end{array}\right] : \quad \begin{pmatrix} u \\ w \end{pmatrix} \xrightarrow{x} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{6}$$

with matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B_u \in \mathbb{R}^{n_x \times n_u}$, $B_w \in \mathbb{R}^{n_x \times n_w}$, $C_y \in \mathbb{R}^{n_y \times n_x}$, $D_{yw} \in \mathbb{R}^{n_y \times n_w}$, $C_z \in \mathbb{R}^{n_z \times n_x}$, $D_{zu} \in \mathbb{R}^{n_z \times n_w}$, where $u$, $w$, $y$, $z$ are state, disturbance, measurement, and controlled vectors. Consider also non-strictly causal dynamical stabilizing output-feedback controller

$$K \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] : \quad y \xrightarrow{h} u, \tag{7}$$

where $h_k \in \mathbb{R}^{n_h}$, and $A_c$, $B_c$, $C_c$, $D_c$ are unknown matrices. The closed-loop system formed by the system (6) and the controller (7) can be represented by

$$F_{cl}(K) \sim \left[\begin{array}{cc|c} A + B_u D_c C_y & B_u C_c & B_w + B_u D_c D_{yw} \\ B_c C_y & A_c & B_c D_{yw} \\ \hline C_z + D_{zu} D_c C_y & D_{zu} C_c & D_{zu} D_c D_{yw} \end{array}\right] : \quad w \xrightarrow{\binom{x}{h}} z. \tag{8}$$

To solve the problem of $\mathcal{H}_2$-optimal control means to find the matrices of the controller (7), such that $\mathcal{H}_2$-norm of the closed-loop system (8) is minimal, i.e. $\|F_{cl}(K)\|_2 \to \min\limits_K$.

It has already been noted that the problem of designing $\mathcal{H}_2$-optimal controller is associated with the problem of input-output invariance of some auxiliary system [2, 3, 6, 9, 10]. To solve the last problem, several additional definitions are introduced, closely related to the concepts of controllable and observable invariants [1, 7]. Namely, for the system $F: w \xrightarrow{x} z$ defined by the quadruple $(A, B, C, D)$ where $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^{n_z}$, we define two sets: $\mathcal{W}(F)$ and $\mathcal{S}(F)$ (see, for example, [8]).

**Definition 4.** The stabilizable weakly unobservable subspace $\mathcal{W}(F)$ is the largest subspace $\mathcal{W} \subseteq \mathbb{R}^{n_x}$ for which there exists a matrix $\Pi$ of suitable dimensions, such that $\mathcal{W} \subseteq \ker(C + D\Pi)$, $(A + B\Pi)\mathcal{W} \subseteq \mathcal{W}$ and $\rho(A + B\Pi) < 1$.

**Definition 5.** The detectable strongly controllable subspace $\mathcal{S}(F)$ is the smallest subspace $\mathcal{S} \subseteq \mathbb{R}^{n_x}$ for which there exists a matrix $\Lambda$ of suitable dimensions, such that $\operatorname{im}(B + \Lambda D) \subseteq \mathcal{S}$, $(A + \Lambda C)\mathcal{S} \subseteq \mathcal{S}$ and $\rho(A + \Lambda C) < 1$.

Let us also introduce two auxiliary matrices $P$ and $Q$ associated with the system (6) as the largest in the sense of the matrix order ($X \succ Y \Leftrightarrow X - Y \succ 0$) matrices-solutions to the inequalities

$$M_1(P) = \begin{bmatrix} A^{\mathrm{T}}PA - P + C_z^{\mathrm{T}}C_z & C_z^{\mathrm{T}}D_{zu} + A^{\mathrm{T}}PB_u \\ D_{zu}^{\mathrm{T}}C_z + B_u^{\mathrm{T}}PA & D_{zu}^{\mathrm{T}}D_{zu} + B_u^{\mathrm{T}}PB_u \end{bmatrix} \succeq 0, \tag{9a}$$

$$M_2(Q) = \begin{bmatrix} AQA^{\mathrm{T}} - Q + B_wB_w^{\mathrm{T}} & B_wD_{yw}^{\mathrm{T}} + AQC_y^{\mathrm{T}} \\ D_{yw}B_w^{\mathrm{T}} + C_yQA^{\mathrm{T}} & D_{yw}D_{yw}^{\mathrm{T}} + C_yQC_y^{\mathrm{T}} \end{bmatrix} \succeq 0. \tag{9b}$$

For a pair $(P, Q)$, we additionally define the matrices $C_P$, $D_P$, $B_Q$ and $D_Q$ in accordance with the formulas

$$\begin{bmatrix} C_P^{\mathrm{T}} \\ D_P^{\mathrm{T}} \end{bmatrix} [C_P \quad D_P] = M_1(P), \qquad \begin{bmatrix} B_Q \\ D_Q \end{bmatrix} [B_Q^{\mathrm{T}} \quad D_Q^{\mathrm{T}}] = M_2(Q), \tag{10}$$

provided that both $[C_P \quad D_P]$ and $[B_Q^{\mathrm{T}} \quad D_Q^{\mathrm{T}}]$ are of full rank.

The solution to the problem of parametrization of $\mathcal{H}_2$-optimal controllers is given in the form of the following theorem.

**Theorem 1** [2, 10]. *For a system* (6), *there exists an $\mathcal{H}_2$-optimal controller of the form* (7) *if and only if the following conditions are satisfied:*

*(i)* $(A, B_u)$ *is stabilizable,*

*(ii)* $(C_y, A)$ *is detectable,*

*(iii)* $\operatorname{im}(B_Q - B_uD_P^+R) \subseteq \mathcal{W}(F_{P_u})$,

*(iv)* $\mathcal{S}(F_{Q_y}) \subseteq \ker(C_P - RD_Q^+C_y)$,

*(v)* $\mathcal{S}(F_{Q_y}) \subseteq \mathcal{W}(F_{P_u})$,

*(vi)* $(A - B_uD_P^+RD_Q^+C_y)\mathcal{S}(F_{Q_y}) \subseteq \mathcal{W}(F_{P_u})$,

*where*

$$R = (D_P^{\mathrm{T}})^+(D_{zu}^{\mathrm{T}}C_zQC_y^{\mathrm{T}} + B_u^{\mathrm{T}}PAQC_y^{\mathrm{T}} + B_u^{\mathrm{T}}PB_wD_{yw}^{\mathrm{T}})(D_Q^{\mathrm{T}})^+, \tag{11}$$

*and the systems $F_{P_u}$ and $F_{Q_y}$ are defined by*

$$F_{P_u} \sim \left[\begin{array}{c|c} A & B_u \\ \hline C_P & D_P \end{array}\right], \quad F_{Q_y} \sim \left[\begin{array}{c|c} A & B_Q \\ \hline C_y & D_Q \end{array}\right]. \tag{12}$$

*If the conditions of the theorem are met, the set of all dynamic $\mathcal{H}_2$-optimal controllers of the form* (7) *is given by*

$$K \sim \left[\begin{array}{cc|c} A + B_u\Pi + \Lambda C_y - B_u\widetilde{D}C_y & B_u\widetilde{C} & B_u\widetilde{D} - \Lambda \\ -\widetilde{B}C_y & \widetilde{A} & \widetilde{B} \\ \hline \Pi - \widetilde{D}C_y & \widetilde{C} & \widetilde{D} \end{array}\right], \tag{13}$$

*where the choice of matrices $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ is limited by the fact that the transfer function*

$$\widetilde{F}(z) = \widetilde{D} + \widetilde{C}(zI_{n_x} - \widetilde{A})^{-1}\widetilde{B} \tag{14}$$

*belongs to the following algebraic sum of spaces: $\widetilde{F}(z) \in N_F + M_F$, where*

$$N_F = \left\{N \in \mathbb{R}^{n_u \times n_y} : D_P N D_Q = -R\right\}, \tag{15a}$$

$$M_F = \left\{M(z) \in \mathcal{RH}_2^{n_u \times n_y} : F_1(z)M(z)F_2(z) = 0\right\}, \tag{15b}$$

*and*

$$F_1(z) = D_P + (C_P + D_P\Pi)(zI_{n_x} - A - B_u\Pi)^{-1}B_u, \tag{16a}$$

$$F_2(z) = D_Q + C_y(zI_{n_x} - A - \Lambda C_y)^{-1}(B_Q + \Lambda D_Q). \tag{16b}$$

A solid analysis of the statement of the theorem can be found in [11]. For the case when left/right-invertible system has no invariant zeros, the similar theorem can be formulated with a certain changes [2]. In this case, the statement will additionally include the condition of *uniqueness* of the $\mathcal{H}_2$-optimal controller if one exists.

## 3. PARAMETRIZATION OF ANISOTROPIC CONTROLLERS

### *3.1. Problem Statement and Solution*

The problem of optimal anisotropic controller design for linear discrete time invariant systems was solved in [13]. The conditions under which the controller was designed ensure the existence and uniqueness of the solution, and the controller itself was specified in a strictly causal form. This section provides a solution to a similar problem, which consists of parameterizing all optimal non-strictly causal anisotropic controllers.

*Problem 1.* For a system (6) driven by external disturbance with the constraint $\overline{\mathbf{A}}(W) \leqslant a$, describe the parametric set of optimal anisotropic controllers of the form (7), i.e. parameterize non-strictly causal stabilizing dynamical controllers minimizing the anisotropic norm of the corresponding closed-loop system.

It is known that when solving the anisotropic analysis and synthesis problems in the optimal setting, it is necessary to consider an additional mathematical construction called the worst-case generating filter. The goal of this filter is to generate the most undesirable (in terms of RMS gain value) external disturbance for a closed-loop system. In accordance with the results obtained in [12, 13], for the systems of the form (2), the worst-case filter is of the form

$$G \sim \left[\begin{array}{c|c} A + BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right]: \quad v \xrightarrow{x} w, \tag{17}$$

where $L \in \mathbb{R}^{n_w \times n_x}$ and $\Sigma \succ 0$ are the matrices chosen to maximize the RMS gain $\|P_{zw}G\|_2/\|G\|_2$ under the constraint $\overline{\mathbf{A}}(W) \leqslant a$. Here and below $V = \{v_k\}_{k\geqslant 0}$ — standard Gaussian white noise.

The main idea of solving the problem 1 is to consider a system formed by a successive connection of the worst-case shaping filter and the original system $F$, and then to parameterize the $\mathcal{H}_2$-optimal controllers for the obtained system. First of all we note that taking into account the shaping filter (17), the system (6) is equivalent from the point of view of the corresponding dynamic processes to the system

$$\overline{F} \sim \left[\begin{array}{c|cc} \overline{A} & \overline{B}_u & \overline{B}_w \\ \hline \overline{C}_y & 0 & \overline{D}_{yw} \\ \overline{C}_z & \overline{D}_{zu} & 0 \end{array}\right] = \left[\begin{array}{c|cc} A' + B'_w L & B'_u & B'_w \Sigma^{1/2} \\ \hline C'_y + D'_{yw} L & 0 & D'_{yw}\Sigma^{1/2} \\ C'_z & D'_{zu} & 0 \end{array}\right] : \begin{pmatrix} \overline{u} \\ v \end{pmatrix} \overset{\binom{x}{h}}{\to} \begin{pmatrix} \overline{y} \\ z \end{pmatrix}, \tag{18}$$

where the new variables are defined as $\overline{u}_k = \begin{pmatrix} u_k^{\mathrm{T}} & h_{k+1}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ and $\overline{y}_k = \begin{pmatrix} y_k^{\mathrm{T}} & h_k^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$; the matrices used in the expression (18) have the following structure:

$$A' = \begin{bmatrix} A & 0 \\ 0 & 0_{n_h \times n_h} \end{bmatrix}, \; B'_u = \begin{bmatrix} B_u & 0 \\ 0 & I_{n_h} \end{bmatrix}, \; B'_w = \begin{bmatrix} B_w \\ 0_{n_h \times n_w} \end{bmatrix}, \tag{19a}$$

$$C'_y = \begin{bmatrix} C_y & 0 \\ 0 & I_{n_h} \end{bmatrix}, \; D'_{yw} = \begin{bmatrix} D_{yw} \\ 0_{n_h \times n_w} \end{bmatrix}, \tag{19b}$$

$$C'_z = [C_z \quad 0_{n_z \times n_h}], \; D'_{zu} = [D_{zu} \quad 0_{n_z \times n_h}]; \tag{19c}$$

matrices $L$ and $\Sigma$ correspond to the shaping filter $G$ (which is the worst-case one for the system (18)) generating a colored signal with the mean anisotropy less or equal to a given threshold $a \geqslant 0$ from standard Gaussian white noise $V = \{v_k\}_{k \geqslant 0}$.

**Theorem 2.** *For a system* (6) *with an external disturbance satisfying the constraint* $\overline{\mathbf{A}}(W) \leqslant a$, *there is an optimal anisotropic controller of the form* (7) *iff the conditions (i)–(vi) of the Theorem 1 hold true. If these conditions are met, the set of all optimal anisotropic controllers of the form* (7) *for the system* (6) *is determined by the formula* $\overline{u}_k = \begin{pmatrix} u_k^{\mathrm{T}} & h_{k+1}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ *where control* $\overline{u}_k$ *is given by the following set of optimal anisotropic controllers for the system* (18):

$$\overline{K} \sim \left[\begin{array}{cc|c} \overline{A} + \overline{B}_u \overline{\Pi} + \overline{\Lambda}\overline{C}_y - \overline{B}_u \widetilde{D}\overline{C}_y & \overline{B}_u \widetilde{C} & \overline{B}_u \widetilde{D} - \overline{\Lambda} \\ -\widetilde{B}\overline{C}_y & \widetilde{A} & \widetilde{B} \\ \hline \overline{\Pi} - \widetilde{D}\overline{C}_y & \widetilde{C} & \widetilde{D} \end{array}\right]. \tag{20}$$

*The matrices* $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$ *correspond to the transfer function*

$$\widetilde{F}(z) = \widetilde{D} + \widetilde{C}(zI_{n_x+n_h} - \widetilde{A})^{-1}\widetilde{B} \tag{21}$$

*belonging to the sum of subspaces* $\widetilde{F}(z) \in N_{\overline{F}} + M_{\overline{F}}$, *where*

$$N_{\overline{F}} = \{\overline{N} \in \mathbb{R}^{(n_u+n_h)\times(n_y+n_h)} : \overline{D}_P \overline{N}\,\overline{D}_Q = -\overline{R}\}, \tag{22a}$$

$$M_F = \{\overline{M}(z) \in \mathcal{RH}_2^{(n_u+n_h)\times(n_y+n_h)} : \overline{F}_1(z)\overline{M}(z)\overline{F}_2(z) = 0\}, \tag{22b}$$

*where*

$$\overline{F}_1(z) = \overline{D}_P + (\overline{C}_P + \overline{D}_P\overline{\Pi})(zI_{n_x+n_h} - \overline{A} - \overline{B}_u\overline{\Pi})^{-1}\overline{B}_u, \tag{23a}$$

$$\overline{F}_2(z) = \overline{D}_Q + \overline{C}_y(zI_{n_x+n_h} - \overline{A} - \overline{\Lambda}\overline{C}_y)^{-1}(\overline{B}_Q + \overline{\Lambda}\overline{D}_Q) \tag{23b}$$

*and*

$$\overline{R} = (\overline{D}_P^{\mathrm{T}})^{+}(\overline{D}_{zu}^{\mathrm{T}}\overline{C}_z\overline{Q}\overline{C}_y^{\mathrm{T}} + \overline{B}_u^{\mathrm{T}}\overline{P}\overline{A}\overline{Q}\overline{C}_y^{\mathrm{T}} + \overline{B}_u^{\mathrm{T}}\overline{P}\overline{B}_w\overline{D}_{yw}^{\mathrm{T}})(\overline{D}_Q^{\mathrm{T}})^{+}. \tag{24}$$

*The matrices* $\overline{C}_P$, $\overline{D}_P$, $\overline{B}_Q$ *and* $\overline{D}_Q$ *are introduced according to* (10) *for the matrices* $M_1(\overline{P})$ *and* $M_2(\overline{Q})$ *associated with the system* (18), *and the matrices* $\overline{\Pi}$ *and* $\overline{\Lambda}$ *relate to the sets* $\mathcal{W}(\overline{F}_{P_u})$ *and* $\mathcal{S}(\overline{F}_{Q_y})$ *introduced according to 4 i 5.*

The proof of the theorem is given in the Appendix.

**Corollary 1.** *If in the Theorem 2 it is also true that the transfer function*

$$F^{ol}_{yw}(z) = D_{yw} + C_y(zI_{n_x} - A)^{-1}B_w \tag{25}$$

*is right invertible, and the transfer function*

$$F^{ol}_{zu}(z) = D_{zu} + C_z(zI_{n_x} - A)^{-1}B_u \tag{26}$$

*is left reversible then the optimal anisotropic controller exists and is unique.*

### *3.2. Numerical Example*

As an example, consider a system of the form (6) with matrices

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{27a}$$

$$C_y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{zu} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{27b}$$

We assume that the external disturbance has the mean anisotropy bounded by a certain number $a \geqslant 0$. Let us set the goal of the example as to solve the problem of parametrization of optimal anisotropic controllers of order not higher than the order of the system itself. Moreover, for the sake of simplicity we will require that the number of additional variables is minimal, i.e., according to (21), $\widetilde{F}(z) = \widetilde{D}$.

Following the required calculations, we can verify that the system (18) has the form

$$\overline{F} \sim \left[\begin{array}{cc|cc|c} A + B_wL_1 & B_wL_2 & B_u & 0_{2\times2} & B_w\sqrt{\sigma} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times1} & I_2 & 0_{2\times1} \\ \hline I_2 & 0_{2\times2} & 0_{2\times1} & 0_{2\times2} & 0_{2\times1} \\ 0_{2\times2} & I_2 & 0_{2\times1} & 0_{2\times2} & 0_{2\times1} \\ \hline C_z & 0_{2\times2} & D_{zu} & 0_{2\times2} & 0_{2\times1} \end{array}\right], \tag{28}$$

and the corresponding matrices $\overline{P}$ and $\overline{Q}$ defined by the formulas (9) are as follows:

$$\overline{P} = \begin{bmatrix} \overline{P}_{11} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad \overline{Q} = \begin{bmatrix} \overline{Q}_{11} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad \overline{Q}_{11} = B_wB_w^{\mathrm{T}}\sqrt{\sigma}, \tag{29}$$

where $\overline{P}_{11}$ is the solution to the Riccati equation

$$\overline{P}_{11} = (A + B_wL_1)^{\mathrm{T}}\overline{P}_{11}(A + B_wL_1) + C_z^{\mathrm{T}}C_z \tag{30a}$$

$$- (A + B_wL_1)^{\mathrm{T}}\overline{P}_{11}B_u(D_{zu}^{\mathrm{T}}D_{zu} + B_u^{\mathrm{T}}\overline{P}_{11}B_u)^{-1}B_u^{\mathrm{T}}\overline{P}_{11}(A + B_wL_1), \tag{30b}$$

where, in order to simplify the further calculations, *we will immediately assume* that $L_2 = 0$ (one can show this statement is true). After this, the matrices $\overline{D}_P$, $\overline{C}_P$, $\overline{D}_Q$ and $\overline{B}_Q$:

$$\overline{D}_P = \begin{bmatrix} (\overline{D}_P)_{11} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} \end{bmatrix} = \begin{bmatrix} (D_{zu}^{\mathrm{T}} D_{zu} + B_u^{\mathrm{T}} \overline{P}_{11} B_u)^{1/2} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} \end{bmatrix}, \tag{31a}$$

$$\overline{C}_P = \begin{bmatrix} (\overline{C}_P)_{11} & 0_{1\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} = \begin{bmatrix} (\overline{D}_P)_{11}^{-1} B_u^{\mathrm{T}} \overline{P}_{11}(A + B_w L_1) & 0_{1\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \tag{31b}$$

$$\overline{D}_Q = \begin{bmatrix} (\overline{D}_Q)_{11} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} = \begin{bmatrix} (B_w B_w^{\mathrm{T}})^{1/2}\sqrt{\sigma} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \tag{31c}$$

$$\overline{B}_Q = \begin{bmatrix} (\overline{B}_Q)_{11} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} = \begin{bmatrix} (A + B_w L_1)(\overline{D}_Q)_{11} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}. \tag{31d}$$

We also calculate the matrix $R$ using (11):

$$\overline{R} = \begin{bmatrix} (\overline{C}_P)_{11}(\overline{D}_Q)_{11} & 0_{1\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}. \tag{32}$$

Now one can check the conditions of the Theorem 2. Obviously, the pair $(\overline{A}, \overline{B}_u)$ is stabilizable, and the pair $(\overline{C}_y, \overline{A})$ is detectable. Now one needs to introduce the sets $\mathcal{W}(\overline{F}_P)$ and $\mathcal{S}(\overline{F}_Q)$. According to the definitions (4) and (5), the matrices $\overline{\Pi}$ and $\overline{\Lambda}$ satisfy the following conditions:

$$\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_{11} & \overline{\Pi}_{12} \\ \overline{\Pi}_{21} & \overline{\Pi}_{22} \end{bmatrix} = \begin{bmatrix} -(\overline{D}_P)_{11}^{-1}(\overline{C}_P)_{11} & 0_{1\times 2} \\ \overline{\Pi}_{21} & \overline{\Pi}_{22} \end{bmatrix}, \quad \rho(\overline{\Pi}_{22}) < 1, \tag{33a}$$

$$\overline{\Lambda} = \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} \\ \overline{\Lambda}_{21} & \overline{\Lambda}_{22} \end{bmatrix} = \begin{bmatrix} -(\overline{B}_Q)_{11}(\overline{D}_Q)_{11}^{-1} & \overline{\Lambda}_{12} \\ 0_{2\times 2} & \overline{\Lambda}_{22} \end{bmatrix}, \quad \rho(\overline{\Lambda}_{22}) < 1. \tag{33b}$$

To simplify the calculations, we choose $\overline{\Pi}_{21} = \overline{\Pi}_{22} = 0_{2\times 2}$ and $\overline{\Lambda}_{12} = \overline{\Lambda}_{22} = 0_{2\times 2}$. One should keep in mind that the particular choice of these matrices leads to a narrowing of the set of the optimal anisotropic controllers. The choice made leads to the fact that $\mathcal{W}(\overline{F}_P) = \mathbb{R}^4$ and $\mathcal{S}(\overline{F}_Q) = \{0\}^4$, after which the conditions *(iii)–(vi)* of the Theorem 2 can be trivially verified.

In this example, a controller with representation (20) under the condition $\widetilde{F}(z) = \widetilde{D}$ is completely determined by the matrix $\widetilde{D}$, satisfying the requirement $\overline{D}_P \widetilde{D} \overline{D}_Q = -\overline{R}$. Substituting the previously found matrices into the last equality, we obtain that $\widetilde{D}$ have the form

$$\widetilde{D} = \begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \\ \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix}, \tag{34}$$

where $\widetilde{D}_{11} = \overline{\Pi}_{11}$, which completes the procedure of describing all optimal anisotropic controllers (20) associated with the relation $\overline{u}_k = \begin{pmatrix} u_k^{\mathrm{T}} & h_{k+1}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$.

Let us make a few important comments.

In order to obtain the final solution to the problem, the resulting system of equations must be supplemented with a system of equations determining the worst-case generating filter, thus finding the variables $L_1 \in \mathbb{R}^{1\times 2}$ and $\sigma > 0$ (see, for example, [12]).

Also, since in the framework of the considered example, the state was observed (measured) precisely, it seems natural to choose a controller in the form of the static state-feedback $u_k = K x_k$. In this case, the optimal choice of matrix $K$ is $K = \widetilde{D}_{11}$.

Let us also present a solution to this problem for $a = 0$ (this case was chosen for simplicity, since there is no need to solve the auxiliary problem associated with the generating filter). It can be shown that all controllers with the representation

$$K \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \approx \left[\begin{array}{cc|cc} 0 & 0 & -1 & 1 \\ -\varkappa - 1.4773 & -\varkappa - 1.4773 & \varkappa + 1 & \varkappa + 2.1823 \\ \hline -\varkappa - 1.4773 & -\varkappa - 1.4773 & \varkappa & \varkappa + 2.1823 \end{array}\right] \tag{35}$$

are optimal, and have the same closed-loop system that does not depend on the specific choice of $\varkappa$ (the choice of which is constrained by the inclusion $\varkappa \in (-2.4773;\ -0.4773)$ providing that the spectral radius of the matrix $A_c$ is less than 1):

$$x_{k+1} \approx \begin{bmatrix} -1 & 1 \\ -0.4773 & 0.7051 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_k, \tag{36a}$$

$$z_k \approx \begin{bmatrix} -1.4773 & 0.7051 \\ 0 & 1 \end{bmatrix} x_k. \tag{36b}$$

Note that $\varkappa \approx -1.4773$ in (35) corresponds to a static state-feedback controller $u_k = D_c x_k$.

## 4. CONCLUSION

The paper provides a parametrization of a set of optimal anisotropic controllers for linear discrete time invariant systems. The results obtained can find application in solving practical problems of navigation and control, in particular, in cases when additional constraints are imposed on the control actions. The results can also be useful to solve the problem of parameterizing a set of suboptimal anisotropic controllers and estimators.

*APPENDIX*

**Proof of Theorem 2.** First let us show that the conditions *(i)–(vi)* from the formulation of the Theorem 2 are equivalent to the following:

*(a)* $(\overline{A}, \overline{B}_u)$ is stabilizable,

*(b)* $(\overline{C}_y, \overline{A})$ is detectable,

*(c)* $\operatorname{im}(\overline{B}_Q - \overline{B}_u \overline{D}_P^+ \overline{R}) \subseteq \mathcal{W}(\overline{F}_{P_u})$,

*(d)* $S(\overline{F}_{Q_y}) \subseteq \ker(\overline{C}_P - \overline{R}\overline{D}_Q^+ \overline{C}_y)$,

*(e)* $\mathcal{S}(\overline{F}_{Q_y}) \subseteq \mathcal{W}(\overline{F}_{P_u})$,

*(f)* $(\overline{A} - \overline{B}_u \overline{D}_P^+ \overline{R}\overline{D}_Q^+ \overline{C}_y)\mathcal{S}(\overline{F}_{Q_y}) \subseteq \mathcal{W}(\overline{F}_{P_u})$

where matrices $\overline{C}_P$, $\overline{D}_P$, $\overline{B}_Q$, $\overline{D}_Q$ and $\overline{R}$, as well as systems $\overline{F}_{P_u}$ and $\overline{F}_{Q_y}$ are set in accordance to the material presented in Section 2.3 in relation to the system (18). Note that the conditions *(a)–(f)* are a direct analogue of the conditions *(i)–(vi)* of the Theorem 1 for system (18).

The equivalence of *(i)* ⇔ *(a)* and *(ii)* ⇔ *(b)* is obvious due to the notation (19). For further proof, let us determine the relation of the sets $\mathcal{W}(F_{P_u})$ and $\mathcal{S}(F_{Q_y})$ from Theorem 1 to the sets $\mathcal{W}(\overline{F}_{P_u})$ and $\mathcal{S}(\overline{F}_{Q_y})$, respectively. Given the system $\overline{F}$, using the Definitions 4 and 5, it can be verified that there exist matrices $\overline{\Pi}$ and $\overline{\Lambda}$ such that

$$\mathcal{W}(\overline{F}_{P_u}) = \mathcal{W}(F_{P_u}) \times \mathbb{R}^{n_h}, \qquad \mathcal{S}(\overline{F}_{Q_y}) = \mathcal{S}(F_{Q_y}) \times \{0\}^{n_h}, \tag{A.1a}$$

$$(\overline{A} + \overline{B}_u \overline{\Pi})\mathcal{W}(\overline{F}_{P_u}) \subseteq \mathcal{W}(\overline{F}_{P_u}), \qquad (\overline{A} + \overline{\Lambda}\overline{C}_y)\mathcal{S}(\overline{F}_{Q_y}) \subseteq \mathcal{S}(\overline{F}_{Q_y}), \tag{A.1b}$$

$$\rho(\overline{A} + \overline{B}_u \overline{\Pi}) < 1, \qquad \rho(\overline{A} + \overline{\Lambda}\overline{C}_y) < 1. \tag{A.1c}$$

After this, we can conclude that the equivalence of the conditions *(iii)*$\Leftrightarrow$*(c)* and *(iv)*$\Leftrightarrow$*(d)* holds due to the fact that

$$\ker(\overline{C}_P - \overline{R}\overline{D}_Q^+\overline{C}_y) = \ker(C_P - RD_Q^+C_y) \times \mathbb{R}^{n_h}, \tag{A.2a}$$

$$\mathrm{im}(\overline{B}_Q - \overline{B}_u\overline{D}_P^+\overline{R}) = \mathrm{im}(B_Q - B_uD_P^+R) \times \{0\}^{n_h}. \tag{A.2b}$$

Finally, by (A.1), the equivalence of the conditions *(v)*$\Leftrightarrow$*(e)* and *(vi)*$\Leftrightarrow$*(e)* is proved.

The structure of the controller (20) is determined by the content of the Theorem 1.

The Theorem 2 is proven.

## ACKNOWLEDGMENTS

The author of the paper thanks Igor Gennadievich Vladimirov and Alexander Viktorovich Yurchenkov who took part in the discussion of the part of the material.

## REFERENCES

1. Basile, G. and Marro, G., Controlled and conditioned invariant subspaces in linear system theory, *J. Optim. Theory Appl.*, 1969, vol. 3, pp. 306–315. https://doi.org/10.1007/BF0093137010.1007/BF00931370
2. Chen, B.M., Saberi, A., and Shamash, Y., Necessary and sufficient conditions under which a discrete time $H_2$-optimal control problem has a unique solution, *Proc. 32nd IEEE Conf. Decision and Control*, 1993, vol. 1, pp. 805–810. https://doi.org/10.1109/CDC.1993.325038
3. Chen, B.M., Saberi, A., Shamash, Y., and Sannuti, P., Construction and parameterisation of all static and dynamic $H_2$-optimal state feedback solutions for discrete time systems, *Proc. 32nd IEEE Conf. Decision and Control*, 1993, vol. 1, pp. 126–131. https://doi.org/10.1109/CDC.1993.325177
4. Diamond, P., Kloeden, P., and Vladimirov, I., Mean anisotropy of homogeneous Gaussian random fields and anisotropic norms of linear translation-invariant operators on multidimensional integer lattices, *J. Appl. Math. Stochast. Anal.*, 2003, vol. 16:3, pp. 209–231. https://doi.org/10.1155/S1048953303000169
5. Doyle, J.C., Glover, K., Khargonekar, P.P., and Francis, B.A., State-space solutions to standard $H_2$ and $H_\infty$ control problems, *IEEE Transactions on Automatic Control*, 1989, vol. 34, pp. 831–847. https://doi.org/10.1109/9.29425
6. Saberi, A., Sannuti, P., and Stoorvogel, A.A., $H_2$ optimal controllers with measurement feedback for continuous-time systems: flexibility in closed-loop pole placement, *Automatica*, 1997, vol. 33, no. 3, pp. 289–304. https://doi.org/10.1016/S0005-1098(96)00195-1
7. Schumacher, J.M., *Dynamic feedback in finite- and infinite-dimensional linear systems*, Mathematisch Centrum, 1981. ISBN: 9061962293.
8. Stoorvogel, A.A., The $H_\infty$ control problem: a state space approach, *Phd Thesis, Mathematics and Computer Science*, Technische Universiteit Eindhoven, 1981. 229 P. https://doi.org/10.6100/IR338287
9. Stoorvogel, A.A., The singular $H_2$ control problem, *Automatica*, 1992, vol. 28, no. 3, pp. 627–631. https://doi.org/10.1016/0005-1098(92)90189-M
10. Trentelman, H.L. and Stoorvogel, A.A., Sampled-data and discrete-time $H_2$ optimal control, *Proc. 32nd IEEE Conf. Decision and Control.*, 1993, vol. 1, pp. 331–336. https://doi.org/10.1109/CDC.1993.325136
11. Trentelman, H.L. and Stoorvogel, A.A., Sampled-data and discrete-time $H_2$ optimal control, *SIAM J. Control and Optimization*, 1995, vol. 33, no. 3, pp. 834–862. https://doi.org/10.1137/S0363012992241995
12. Vladimirov, I.G., Kurdjukov, A.P., and Semyonov, A.V., On computing the anisotropic norm of linear discrete-time-invariant systems, *IFAC Proceedings Volumes*, 1996, vol. 29, no. 1, pp. 3057–3062. https://doi.org/10.1016/S1474-6670(17)58144-6
13. Vladimirov, I.G., Kurdyukov, A.P., and Semyonov, A.V., State-space solution to anisotropy-based stochastic $H\infty$-optimization problem, *IFAC Proceedings Volumes*, 1996, vol. 29, no. 1, pp. 3816–3821.

*This paper was recommended for publication by V.M. Glumov, a member of the Editorial Board*